USA Mathematical Talent Search
Solutions to Problem 2/2/19
www.usamts.org

2/2/19. Let x, y, and z be complex numbers such that $x + y + z = x^5 + y^5 + z^5 = 0$ and $x^3 + y^3 + z^3 = 3$. Find all possible values of $x^{2007} + y^{2007} + z^{2007}$.

Comments There are different ways to appoach this problem. The solution below uses a substitution to eliminate one of the variables, and determines the values of x^3, y^3, and z^3 directly. *Solutions edited by Naoki Sato.*

Solution by: Kristin Cordwell (11/NM)

First, $x+y+z = 0$, so $x+y = -z$. We then cube both sides to get $x^3+3x^2y+3y^2x+y^3 = -z^3$. We rearrange the equation to get $x^3 + y^3 + z^3 = -3x^2y - 3y^2x$. We know that $x^3 + y^3 + z^3 = 3$, so we get $-3xy(x + y) = 3$, or $xy(x + y) = -1$. This also tells us that neither xy nor $x + y$ can be equal to 0.

Now we take the fifth power of $x + y = -z$ to get

$$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = -z^5.$$

Rearranging the equation gives us

$$x^5 + y^5 + z^5 = -5xy(x^3 + 2x^2y + 2xy^2 + y^3).$$

We know that $x^5 + y^5 + z^5 = 0$, so we get $-5xy(x^3 + 2x^2y + 2xy^2 + y^3) = 0$, or

$$xy(x^3 + 2x^2y + 2xy^2 + y^3) = 0.$$

Also, we know that $xy \neq 0$, so we get

$$x^3 + 2x^2y + 2xy^2 + y^3 = x^3 + y^3 + 2xy(x + y) = 0.$$

We know that $xy(x + y) = -1$, so this simplifies as $x^3 + y^3 = 2$. Finally, we know that $x^3 + y^3 + z^3 = 3$, so z^3 must equal 1. By symmetry, $x^3 = y^3 = 1$. Since 2007 is divisible by 3, $x^{2007} + y^{2007} + z^{2007} = 3$.

To show that this value is possible, let $x = 1$, $y = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, which are the cube roots of unity. Then $x + y + z = x^5 + y^5 + z^5 = 0$, and $x^3 + y^3 + z^3 = x^{2007} + y^{2007} + z^{2007} = 3$.