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Solutions to Problem 2/4/18
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2/4/18. For how many integers n between 1 and 10^{2007}, inclusive, are the last 2007 digits of n and n^3 the same? (If n or n^3 has fewer than 2007 digits, treat it as if it had zeros on the left to compare the last 2007 digits.)

Credit This problem was proposed by Paul Bateman, Professor Emeritus at the University of Illinois at Urbana-Champaign.

Comments When solving a congruence with modulus m, we can look at the congruence with respect to each prime factor of m. Then, the solutions can be sewn together using the Chinese Remainder Theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (12/NJ)

Saying that the last 2007 digits of n and n^3 are the same is equivalent to saying that $n \equiv n^3 \pmod{10^{2007}}$, or

$$n(n-1)(n+1) \equiv 0 \pmod{10^{2007}}.$$

Therefore, $n(n-1)(n+1)$ is divisible by both 5^{2007} and 2^{2007}

Since only one of n, $n-1$, and $n+1$ can be divisible by 5, whichever one is divisible by 5 must also be divisible by 5^{2007}, so

$$n \equiv 0,\ 1,\ \text{or}\ -1 \pmod{5^{2007}}.$$

Similarly, if n is even, then both $n-1$ and $n+1$ are odd, so $n \equiv 0 \pmod{2^{2007}}$. On the other hand, if n is odd, then $(n-1)(n+1) \equiv 0 \pmod{2^{2007}}$. However, the difference between $n-1$ and $n+1$ is 2, so only one of them can be divisible by 4. Call this one $n \pm 1$. Hence, $n \mp 1$ is divisible by 2 but not 4. Therefore, $n \pm 1$ must be divisible by 2^{2006} in order that $(n-1)(n+1) \equiv 0 \pmod{2^{2007}}$, so $n \equiv \pm 1 \pmod{2^{2006}}$. Hence, if n is odd,

$$n \equiv 1,\ 2^{2006}-1,\ 2^{2006}+1,\ \text{or}\ 2^{2007}-1 \pmod{2^{2007}}.$$

Recall that if n is even, then $n \equiv 0 \pmod{2^{2007}}$.

There are three possible values of n modulo 5^{2007} and five possible values of n modulo 2^{2007}. By the Chinese Remainder Theorem, there are 15 possible values of n modulo 10^{2007}, which means there are 15 solutions n for $1 \le n \le 10^{2007}$.