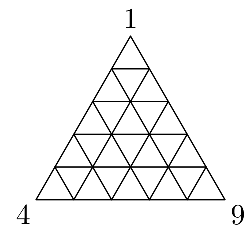
3/1/18.

(a) An equilateral triangle is divided into 25 congruent smaller equilateral triangles, as shown. Each of the 21 vertices is labeled with a number such that for any three consecutive vertices on a line segment, their labels form an arithmetic sequence. The vertices of the original equilateral triangle are labeled 1, 4, and 9. Find the sum of the 21 labels.



(b) Generalize part (a) by finding the sum of the labels when there are n^2 smaller congruent equilateral triangles, and the labels of the original equilateral triangle are a, b, and c.

Credit This problem was proposed by Naoki Sato.

Comments The desired sums can be systematically computed by recognizing that for any consecutive vertices lying on a line, the labels form an arithmetic sequence. A symmetry argument can also be used to give a quick solution. *Solutions edited by Naoki Sato.*

Solution 1 by: Erik Madsen (12/CA)

This solution develops a general equation for part (b) and then applies it to solve part (a).

Without loss of generality, label the top vertex of the triangle a, the lower-left vertex b, and the lower-right vertex c. Now note that in a triangle composed of n^2 smaller triangles, we have $n+1$ horizontal rows of vertices – the first row has 1 vertex, the second has 2, and so on, with the bottom row having $n+1$ vertices. The conditions of the problem indicate that the vertices of each of these horizontal rows form an arithmetic sequence (since if any three consecutive vertices form an arithmetic sequence, then any number of consecutive vertices form an arithmetic sequence), as do the vertices of each of the diagonal edges of the triangle.

Therefore, the value of the left vertex in the m^{th} row is

$$l_m = a + \frac{b-a}{n}(m-1),$$

with m ranging from 1 to $n+1$. Similarly, the value of the right vertex in the m^{th} row is

$$r_m = a + \frac{c-a}{n}(m-1),$$

with m ranging from 1 to $n+1$. Using the formula for the sum of an arithmetic series, the sum of the values of the vertices in row m is

$$s_m = (l_m + r_m) \cdot \frac{m}{2},$$

where m ranges from 1 to $n+1$.

To find the sum S of all vertices of the triangle, we must sum s_m over all m:

$$\begin{aligned}
S &= \sum_{m=1}^{n+1} s_m = \sum_{m=1}^{n+1} (l_m + r_m) \cdot \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left[a + \frac{b-a}{n}(m-1) + a + \frac{c-a}{n}(m-1)\right] \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left[2a + \frac{b+c-2a}{n}(m-1)\right] \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left(a - \frac{b+c-2a}{2n}\right) m + \sum_{m=1}^{n+1} \left(\frac{b+c-2a}{2n}\right) m^2 \\
&= \left(a - \frac{b+c-2a}{2n}\right) \frac{(n+1)(n+2)}{2} + \left(\frac{b+c-2a}{2n}\right) \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \left[a - \frac{b+c-2a}{2n} + \left(\frac{b+c-2a}{2n}\right)\left(\frac{2n+3}{3}\right)\right] \frac{(n+1)(n+2)}{2} \\
&= \left[a + \left(\frac{b+c-2a}{2n}\right)\left(\frac{2n+3}{3} - 1\right)\right] \frac{(n+1)(n+2)}{2} \\
&= \left[a + \left(\frac{b+c-2a}{2n}\right) \cdot \frac{2n}{3}\right] \frac{(n+1)(n+2)}{2} \\
&= \left(a + \frac{b+c-2a}{3}\right) \frac{(n+1)(n+2)}{3} \\
&= \frac{(a+b+c)(n+1)(n+2)}{6}.
\end{aligned}$$

Therefore,

$$S(n,a,b,c) = \frac{(a+b+c)(n+1)(n+2)}{6}.$$

Applying this formula to part (a), we have $S(5,1,4,9) = 98$.

Solution 2 by: James Sundstrom (12/NJ)

(a) Consider two triangles, both divided into 25 smaller vertices and labeled such that the labels of any three consecutive vertices on a line segment form an arithmetic sequence. Obtain a third labeled triangle by adding the corresponding labels of the original two triangles (call this operation addition of labeled triangles). The new triangle will also have the property that the labels of any three consecutive vertices on a line segment form an arithmetic sequence, because, if a_1, b_1, c_1 and a_2, b_2, c_2 are both arithmetic sequences, then

$$\begin{aligned}(c_1 + c_2) - (b_1 + b_2) &= (c_1 - b_1) + (c_2 - b_2) \\ &= (b_1 - a_1) + (b_2 - a_2) \\ &= (b_1 + b_2) - (a_1 + a_2),\end{aligned}$$

and so $a_1 + a_2$, $b_1 + b_2$, $c_1 + c_2$ is also an arithmetic sequence. It is easy to see that the sum of the labels of the new triangle is equal to the sum of the sums of the labels of the original triangles.

Say that a triangle is labeled with (a, b, c) if a is at the top vertex, b is at the bottom left vertex, and c is at bottom right vertex, and all other labels follow the arithmetic sequence rule. Add the three triangles labeled with $(1, 4, 9)$, $(4, 9, 1)$, and $(9, 1, 4)$. The result is the triangle labeled with $(14, 14, 14)$. Since, by symmetry, the original three triangles must have the same label sum, the label sum of this triangle is equal to three times the label sum of the triangle labeled with $(1, 4, 9)$. All the labels of the triangle labeled with $(14, 14, 14)$ are the same, and since the triangle in question has 21 vertices, its label sum is $21 \times 14 = 294$. Therefore the label sum of the triangle labeled with $(1, 4, 9)$ is $294/3 = 98$.

(b) Let $f_n(a, b, c)$ denote the sum of the labels of a triangle labeled with (a, b, c) when there are n^2 smaller triangles. The function f_n is additive, i.e.

$$f_n(a_1, b_1, c_1) + f_n(a_2, b_2, c_2) = f_n(a_1 + a_2, b_1 + b_2, c_1 + c_2),$$

as noted in part (a). By symmetry, $f_n(a, b, c) = f_n(b, c, a) = f_n(c, a, b)$, so

$$3f_n(a, b, c) = f_n(a, b, c) + f_n(b, c, a) + f_n(c, a, b) = f_n(a + b + c, a + b + c, a + b + c).$$

Let T_n denote the n^{th} triangular number. Then all T_{n+1} labels of the triangle labeled with $(a+b+c, a+b+c, a+b+c)$ are the same, namely $a+b+c$, so $f_n(a+b+c, a+b+c, a+b+c) = T_{n+1} \cdot (a + b + c)$. Therefore,

$$\begin{aligned}f_n(a, b, c) &= \frac{1}{3} f_n(a + b + c, a + b + c, a + b + c) \\ &= \frac{1}{3} T_{n+1}(a + b + c) \\ &= \frac{(n + 1)(n + 2)(a + b + c)}{6}.\end{aligned}$$