4/3/19. Prove that 101 divides infinitely many of the numbers in the set

$$\{2007, 20072007, 200720072007, 2007200720072007, \ldots\}.$$

Comments There are several possible approaches, but the following solution elegantly reduces the elements of the set modulo 101 to a simple formula. *Solutions edited by Naoki Sato.*

Solution by: Daniel Tsai (12/NJ)

Modulo 101, we have $10^0 \equiv 1$, $10^1 \equiv 10$, $10^2 \equiv 100$, $10^3 \equiv 91$, and $10^4 \equiv 1$. Therefore, modulo 101, for each integer $k \geq 0$, $10^{4k} \equiv 1$, $10^{4k+1} \equiv 10$, $10^{4k+2} \equiv 100$, and $10^{4k+3} \equiv 91$. Consequently, for each positive integer n,

$$20072007 \cdots 2007 \equiv (2 \cdot 91 + 0 \cdot 100 + 0 \cdot 10 + 7 \cdot 1)n \equiv 189n \equiv 88n \pmod{101},$$

where the left-hand-side consists of n 2007s. For each positive multiple n of 101, $88n \equiv 0 \pmod{101}$, thus 101 divides infinitely many numbers in the set

$$\{2007, 20072007, 200720072007, 2007200720072007, \ldots\}.$$