USA Mathematical Talent Search

Solutions to Problem 5/1/18

www.usamts.org

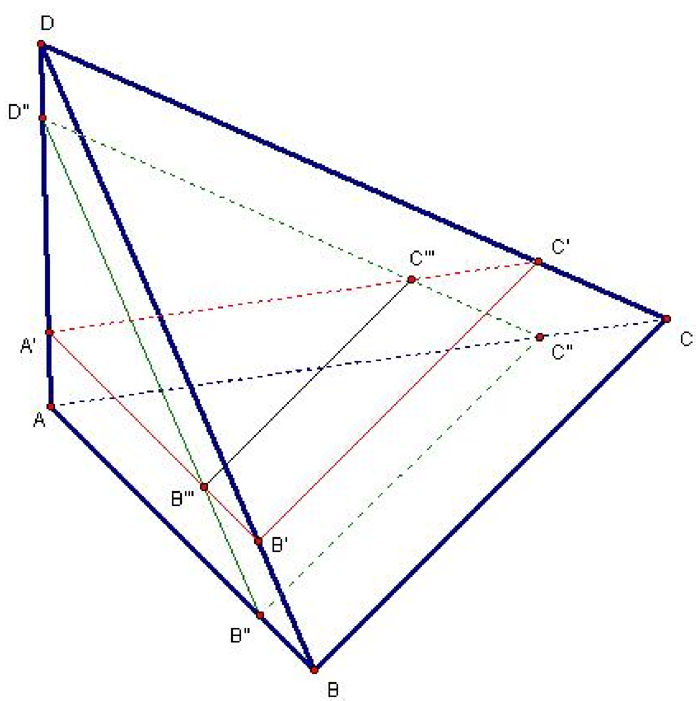
5/1/18. $ABCD$ is a tetrahedron such that $AB = 6$, $BC = 8$, $AC = AD = 10$, and $BD = CD = 12$. Plane $\mathcal{P}$ is parallel to face ABC and divides the tetrahedron into two pieces of equal volume. Plane $\mathcal{Q}$ is parallel to face DBC and also divides $ABCD$ into two pieces of equal volume. Line ℓ is the intersection of planes $\mathcal{P}$ and $\mathcal{Q}$. Find the length of the portion of ℓ that is inside $ABCD$.

Credit This problem was proposed by Richard Rusczyk.

Comments This problem is best solved by using similar tetrahedra, and drawing a nice diagram. To solve three-dimensional geometry problems, one technique that may help is to consider the two-dimenaional analogue. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (12/NJ)

Let A' denote the intersection of plane $\mathcal{P}$ and $\overline{AD}$, and define points B' and C' similarly. Let B'' denote the intersection of plane $\mathcal{Q}$ and $\overline{AB}$, and define points C'' and D'' similarly. Let B''' denote the intersection of $\mathcal{P}$, $\mathcal{Q}$, and face ABD, and let C''' denote the intersection of $\mathcal{P}$, $\mathcal{Q}$, and face ACD. Then the problem asks for the length of $\overline{B'''C'''}$.



Tetrahedrons $ABCD$ and $A'B'C'D$ are similar because plane $\mathcal{P}$ is parallel to face ABC. The volume of $ABCD$ is twice the volume of $A'B'C'D$, so $A'D = AD/\sqrt[3]{2} = 10/\sqrt[3]{2}$. Similarly, $AD'' = 10/\sqrt[3]{2}$. Since $A'D+AD'' = AD+A'D''$, we find that $A'D'' = 20/\sqrt[3]{2}-10 = 10(\sqrt[3]{4}-1)$.

Since plane $\mathcal{P}$ and face ABC are parallel, and plane $\mathcal{Q}$ and face DBC are parallel, tetrahedrons $ABCD$ and $A'B'''C'''D''$ are similar. Therefore,

$$B'''C''' = \frac{BC \cdot A'D''}{AD} = \frac{8 \cdot 10(\sqrt[3]{4}-1)}{10} = 8(\sqrt[3]{4}-1).$$