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Solutions to Problem 5/4/19

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5/4/19. Let $P_1P_2P_3 \cdots P_{13}$ be a regular 13-gon. For $1 \le i \le 6$, let $d_i = P_1P_{i+1}$. The 13 diagonals of length d_6 enclose a smaller regular 13-gon, whose side length we denote by s. Express s in the form

$$s = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4 + c_5d_5 + c_6d_6,$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are integers.

Comments In geometry, when trying to find relationships between lengths, it is often useful to find line segments that add up to the lengths in question. The following solution constructs these line segments by using the symmetry of the regular 13-gon. *Solutions edited by Naoki Sato.*

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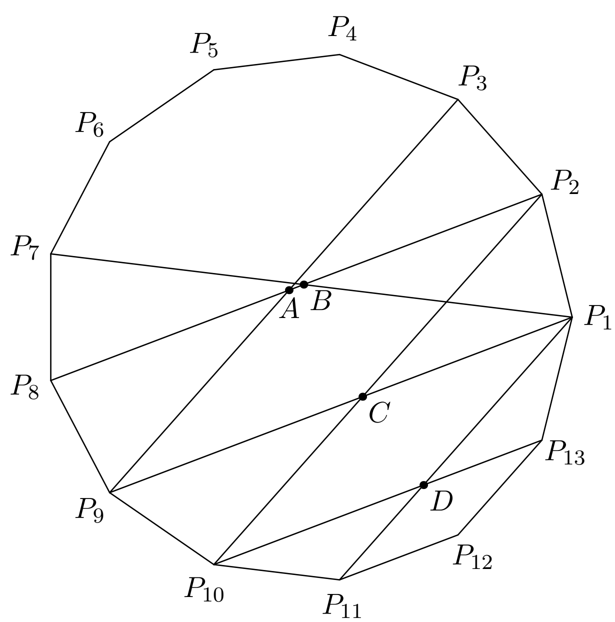

Let A be the intersection of P_2P_8 and P_3P_9, B the intersection of P_1P_7 and P_2P_8, C the intersection of P_1P_9 and P_2P_{10}, and D the intersection of P_1P_{11} and $P_{10}P_{13}$. Since P_1P_7, P_2P_8, and P_3P_9 are all diagonals of length d_6, AB is a side of the smaller regular 13-gon, so $s = AB$.

By symmetry, $BP_2 = AP_8$. Let $x = BP_2 = AP_8$. Since $P_2P_8 = P_1P_7 = d_6$, $AP_2 = d_6 - x$.

Diagonals P_3P_9 and P_2P_{10} are parallel, and diagonals P_2P_8 and P_1P_9 are parallel, so quadrilateral P_2AP_9C is a parallelogram. Hence, $AP_2 = d_6 - x = P_9C$. Since $P_1P_9 = P_1P_6 = d_5$, $P_9C = d_6 - x = d_5 - P_1C$.

Since P_2P_{10} and P_1P_{11} are parallel and P_1P_9 and $P_{13}P_{10}$ are parallel, quadrilateral $P_1CP_{10}D$ is a parallelogram. Hence, $P_1C = DP_{10}$ and we can rewrite the equation above as $d_6 - x = d_5 - DP_{10}$. Since $P_{10}P_{13} = P_1P_4 = d_3$, $d_6 - x = d_5 - (d_3 - DP_{13})$.

Finally, since P_1P_{11} and $P_{13}P_{12}$ are parallel and $P_{13}P_{10}$ and $P_{12}P_{11}$ are parallel, quadrilateral $P_{13}DP_{11}P_{12}$ is a parallelogram. Hence, $DP_{13} = P_{11}P_{12}$ and we can rewrite the equation above as $d_6 - x = d_5 - (d_3 - P_{11}P_{12})$. Since $P_{11}P_{12} = P_1P_2 = d_1$, $d_6 - x = d_5 - (d_3 - d_1)$.

Rearranging the last equation, we find that $x = d_6 - d_5 + d_3 - d_1$. Since $s = d_6 - 2x$, we have

$$s = d_6 - 2(d_6 - d_5 + d_3 - d_1) = 2d_1 - 2d_3 + 2d_5 - d_6.$$