USA Mathematical Talent Search

Round 2 Solutions

Year 25 — Academic Year 2013–2014

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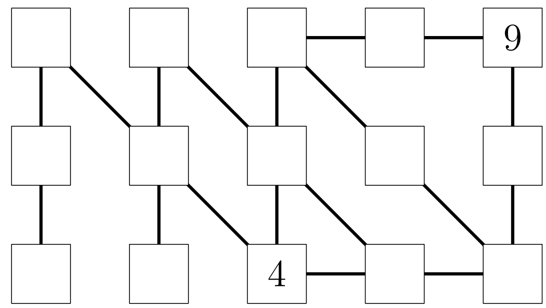
1/2/25. In the 3×5 grid shown, fill in each empty box with a two-digit positive integer so that:

(a) no number appears in more than one box, and

(b) for each of the 9 lines in the grid consisting of three boxes connected by line segments, the box in the middle of the line contains the least common multiple of the numbers in the other two boxes on the line.

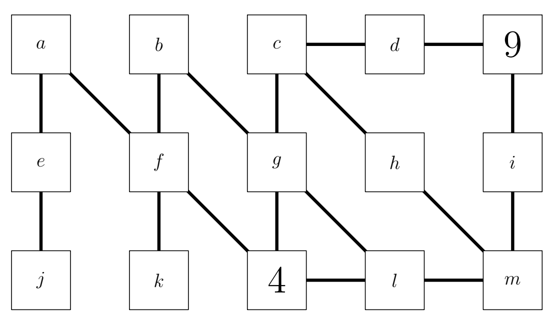


You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

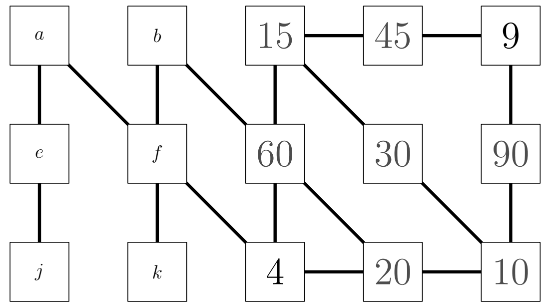
Solution

Let x, y, z be three numbers on a line in that order, so that $y = \text{lcm}(x, z)$. Notice that if x is a divisor of z, then $y = \text{lcm}(x, z) = z$. But we must have $y \neq z$ since all the numbers must be distinct, so we conclude that x cannot be a divisor of z. By the same reasoning, z cannot be a divisor of x. That is, neither number at the end of a line can be a divisor of the number at the other end. Call this the *anti-divisor property.*

Label the unknown numbers in the grid with the variables a through m as shown at right. By the anti-divisor property, m is not a multiple of 4 or 9. But if m's only prime divisors are 2 and/or 3, then $m \leq 6$, which contradicts the requirement that m is a 2-digit number. Therefore, m must have a prime divisor p with $p \geq 5$. Then, h, i, and l are all multiples of m, so they each are multiples of p. g is also a multiple of h, so it is a multiple of p. Since $\text{lcm}(4, c) = g$ and p divides 4 but not g, p must also divide c. Then, d is a multiple of c, so d is a multiple of p. In summary, all of c, d, g, h, i, l, m are multiples of p.

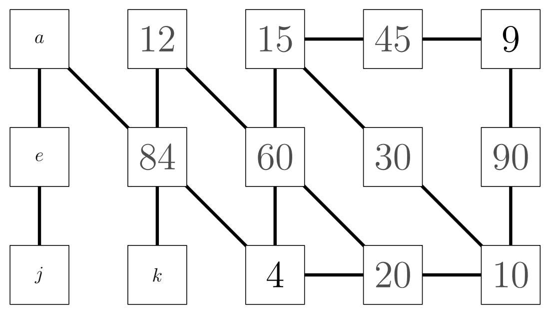


In particular, d and i are each multiples of $9p$, so we must have $p = 5$, because if $p \geq 7$ then there is at most 1 two-digit multiple of $9p$. Hence d and i are 45 or 90 in either order. This means that c and m are each multiples of 5 but divisors of 90, so they must be 10, 15, or 30. However, if one of them were 30, then the line c—h—m would violate the anti-divisor property, so c and m must be 10 and 15 in either order. But if $c = 10$ and $m = 15$, this makes $l = 60$ and $g = 20$, a contradiction since $g \geq l$. Thus, $c = 15$ and $m = 10$.



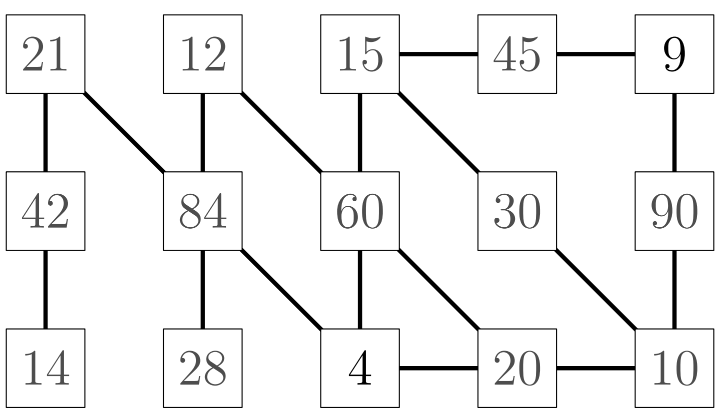
We can now fill in all of the rightmost three columns of the grid, as shown to the right.

Next, we see that b must be a two-digit multiple of 3 but a divisor of 60. The only possibility not yet used in the grid is $b = 12$. Thus f is a multiple of 12, but it cannot be a multiple of 8 because then a would be a multiple of 8, contradicting the anti-divisor property on the line a—f—4. This leaves the only possibilities for f, after also excluding the multiples of 12 already in the grid, as 36 or 84. But a and k must both be two-digit divisors of f, and the only two-digit proper divisors of 36 are 12 and 18, and 12 is already in the grid (at b). Thus we must have $f = 84$.



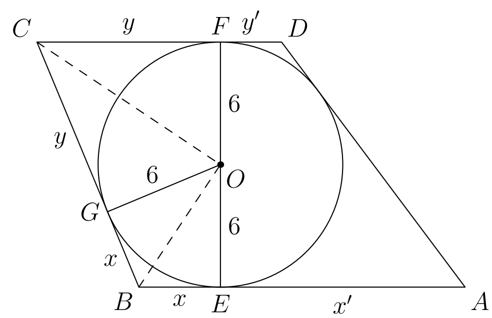
The grid is now as shown at right. Next, we see that a must be a multiple of 21 and a proper divisor of 84, so it must be 21 or 42. But $a = 42$ leaves no value for e, so we must have $a = 21$. Then e must be 42 or 63, but $e = 63$ leaves no two-digit value for j, so we must have $e = 42$, which forces $j = 14$. Finally, k is a two-digit proper divisor of 84 not yet used in the grid, and the only value remaining is $k = 28$.

The completed grid is shown below, and our argument above proves that this solution is unique.



2/2/25. Let $ABCD$ be a quadrilateral with $\overline{AB} \parallel \overline{CD}$, $AB = 16$, $CD = 12$, and $BC < AD$. A circle with diameter 12 is inside of $ABCD$ and tangent to all four sides. Find BC.

Solution



Let O be the center of the circle, let E and F be the points of tangency to $\overline{AB}$ and $\overline{CD}$, respectively, and let G be the point of tangency to $\overline{BC}$, as in the picture above.

Let $x = EB$, and note that since $\triangle OEB \cong \triangle OGB$, we also have $BG = x$. Let $y = FC$, and note that since $\triangle OFC \cong \triangle OGC$, we also have $CG = y$. Further, notice that

$$\angle EOB = \frac{1}{2}\angle EOG = \frac{1}{2}(180^\circ - \angle FOG) = 90^\circ - \frac{1}{2}\angle FOG = 90^\circ - \angle FOC.$$

Thus, $\angle EOB$ and $\angle FOC$ are complementary angles, and hence $\triangle EOB$ and $\triangle FCO$ are similar right triangles. Therefore, $\frac{EB}{EO} = \frac{FO}{FC}$, giving $\frac{x}{6} = \frac{6}{y}$, or $xy = 36$.

Let $x' = EA$ and $y' = FD$. Using the same reasoning as above in trapezoid $AEFD$, we have $x'y' = 36$.

We also have the equations

$$x + x' = 16, \tag{1}$$
$$y + y' = 12. \tag{2}$$

Using $y = \frac{36}{x}$ and $y' = \frac{36}{x'}$, (2) becomes

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{3}. \tag{3}$$

But (3) is equivalent $\dfrac{x + x'}{xx'} = \dfrac{1}{3}$, which simplifies to $3(x + x') = xx'$, so using (1) we have

$$xx' = 48. \tag{4}$$

Now, equations (1) and (4) together tell us that x and x' are the roots of the quadratic equation $t^2 - 16t + 48 = 0$, which factors as $(t - 12)(t - 4) = 0$, so x and x' are 4 and 12 (in some order). Using $xy = x'y' = 36$, we find that the two possible solutions are

$$(x, y, x', y') = (4, 9, 12, 3) \qquad \text{or} \qquad (x, y, x', y') = (12, 3, 4, 9).$$

However, recall that $BC = x + y$ and $AD = x' + y'$, and we must have $BC < AD$. This forces the first solution for (x, y, x', y'), giving a final answer of $BC = 13$.

3/2/25. For each positive integer $n \geq 2$, find a polynomial $P_n(x)$ with rational coefficients such that $P_n\left(\sqrt[n]{2}\right) = \dfrac{1}{1+\sqrt[n]{2}}$. (Note that $\sqrt[n]{2}$ denotes the positive n^{th} root of 2.)

Solution

We use the factorization

$$\frac{1-r^k}{1-r} = 1 + r + r^2 + \cdots + r^{k-1}.$$

Let $k = 2n$ and $r = -\sqrt[n]{2}$. Then the above equation becomes

$$\frac{1-4}{1+\sqrt[n]{2}} = (-\sqrt[n]{2})^{2n-1} + (-\sqrt[n]{2})^{2n-2} + (-\sqrt[n]{2})^{2n-3} + (-\sqrt[n]{2})^{2n-4} + \cdots + (-\sqrt[n]{2}) + 1.$$

Therefore,

$$\frac{1}{1+\sqrt[n]{2}} = \frac{(\sqrt[n]{2})^{2n-1} - (\sqrt[n]{2})^{2n-2} + (\sqrt[n]{2})^{2n-3} - (\sqrt[n]{2})^{2n-4} + \cdots + \sqrt[n]{2} - 1}{3}.$$

Hence, we may choose

$$P_n(x) = \frac{x^{2n-1} - x^{2n-2} + x^{2n-3} - x^{2n-4} + \cdots + x - 1}{3} = \sum_{i=0}^{2n-1} \frac{(-1)^{i+1}}{3} x^i.$$

4/2/25. An infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ is called *spooky* if $a_1 = 1$ and for all integers $n > 1$,

$$\begin{aligned} na_1 + \ (n-1)a_2 + \ (n-2)a_3 + \cdots + \ 2a_{n-1} + a_n &< 0, \\ n^2a_1 + (n-1)^2a_2 + (n-2)^2a_3 + \cdots + 2^2a_{n-1} + a_n &> 0. \end{aligned}$$

Given any spooky sequence $a_1, a_2, a_3, \ldots$, prove that

$$2013^3a_1 + 2012^3a_2 + 2011^3a_3 + \cdots + 2^3a_{2012} + a_{2013} < 12345.$$

Solution

Define the following quantities for each positive integer k:

$$\begin{aligned} t_k &= ka_1 + (k-1)a_2 + \cdots + 2a_{k-1} + a_k, \\ u_k &= k^2a_1 + (k-1)^2a_2 + \cdots + 2^2a_{k-1} + a_k, \\ v_k &= k^3a_1 + (k-1)^3a_2 + \cdots + 2^3a_{k-1} + a_k. \end{aligned}$$

Note that from the given conditions, $t_1 = u_1 = 1$, and $t_k < 0 < u_k$ for all $k > 1$.

We compute:

$$\begin{aligned} t_k + t_{k+1} + \cdots + t_1 = {} & (k + (k-1) + \cdots + 1)a_1 \\ & + ((k-1) + (k-2) + \cdots + 1)a_2 \\ & + ((k-2) + (k-3) + \cdots + 1)a_3 \\ & + \vdots \\ & + a_k. \end{aligned}$$

By using the fact that $1 + 2 + \cdots + n = \frac{n^2+n}{2}$, we have:

$$\begin{aligned} t_k + t_{k-1} + \cdots + t_1 &= \frac{k^2+k}{2}a_1 + \frac{(k-1)^2 + (k-1)}{2}a_2 + \frac{(k-2)^2 + (k-2)}{2}a_3 + \cdots + a_k \\ &= \frac{1}{2}(u_k + t_k). \end{aligned}$$

Thus,

$$u_k = 2(t_1 + t_2 + t_3 + \cdots + t_{k-1}) + t_k.$$

In particular, note that $u_k < 2t_1 = 2$ for all $k > 1$.

Similarly, we compute:

$$\begin{aligned} u_k + u_{k+1} + \cdots + u_1 = {} & (k^2 + (k-1)^2 + \cdots + 1^2)a_1 \\ & + ((k-1)^2 + (k-2)^2 + \cdots + 1^2)a_2 \\ & + ((k-2)^2 + (k-3)^2 + \cdots + 1^2)a_3 \\ & + \vdots \\ & + a_k. \end{aligned}$$

By using the fact that $1^2 + 2^2 + \cdots + n^2 = \frac{2n^3+3n^2+n}{6}$, we have:

$$\begin{aligned} u_k + u_{k-1} + \cdots + u_1 &= \frac{2k^3 + 3k^2 + k}{6} a_1 + \frac{2(k-1)^3 + 3(k-1)^2 + (k-1)}{6} a_2 + \cdots + a_k \\ &= \frac{1}{6}(2v_k + 3u_k + t_k). \end{aligned}$$

Thus,

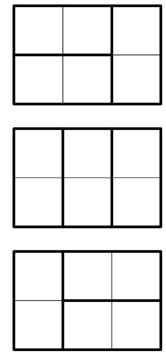
$$v_k = 3(u_k + u_{k-1} + \cdots + u_1) - \frac{3u_k + t_k}{2} = 3(u_{k-1} + \cdots + u_1) + \frac{3u_k - t_k}{2}.$$

Since $u_k < 2$ and $t_k < 0$ for all $k > 1$, we have

$$v_k < 3(2(k-2) + 1) + \frac{3(2)}{2} = 6k - 6.$$

In particular, $v_{2013} < 6(2013) - 6 = 12072 < 12345$, as desired.

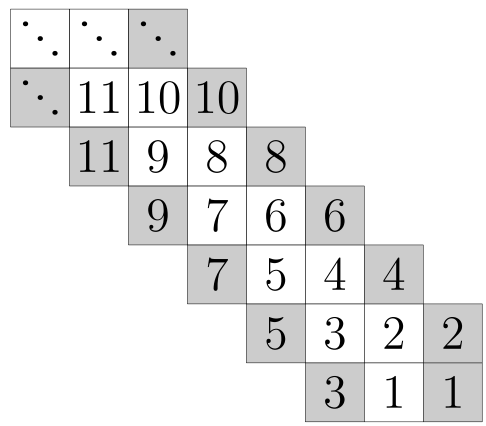
5/2/25. Let S be a planar region. A *domino-tiling* of S is a partition of S into 1×2 rectangles. (For example, a 2×3 rectangle has exactly 3 domino-tilings, as shown to the right.) The rectangles in the partition of S are called *dominoes*.



(a) For any given positive integer n, find a region S_n with area at most $2n$ that has exactly n domino-tilings.

(b) Find a region T with area less than 50000 that has exactly 100002013 domino-tilings.

Solution

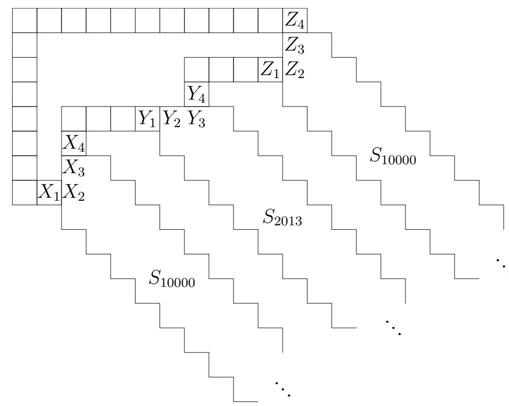


(a) Consider the infinitely repeating pattern of squares shown in the diagram to the right, where we have colored some squares gray and the rest white. For each positive integer n, define S_n to be the region containing the squares in this figure numbered 1 to n inclusive. We claim that S_n has exactly n domino-tilings, which we will prove below. Since S_n has area $2n$, these are our desired regions.

First, for $n \geq 2$, define B_n to be the same region as S_n except omitting the two white squares labeled n and $n-1$. We prove that B_n has a unique domino-tiling, by induction on n. The base case $n = 2$ is clear: B_2 consists only of the gray squares labeled 1 and 2. For the inductive step, let $k \geq 2$ be given and assume that B_k has a unique domino-tiling, and consider B_{k+1}. The gray square labeled $k+1$ touches only the white square labeled $k-1$, so in any domino-tiling of B_{k+1} those two squares must be part of the same piece of the partition (i.e. must be covered by the same domino). But what remains is B_k, which by inductive hypothesis has a unique domino-tiling. Thus, B_{k+1} has a unique domino-tiling.

Now we prove our claim that S_n has exactly n domino-tilings, by induction on n. The base cases of $n = 1$ and $n = 2$ are easily checked: S_1 has a unique domino-tiling since it has only two squares, and S_2 is a 2×2 square, so the only possible domino-tilings are either two horizontal or two vertical dominos, yielding exactly 2 domino-tilings. Now let $k \geq 2$ be a given positive integer, and assume that S_k has exactly k domino-tilings. Consider S_{k+1}, and in particular the domino including the white square labeled $k+1$. If this domino also includes the white square labeled k, then what remains is B_{k+1}, and thus there is only one way to complete the domino-tiling of S_{k+1}. Otherwise, our initial domino covers both squares labeled $k+1$. Then, the region that remains is S_k, so by the inductive hypothesis, there are k ways to complete the domino-tiling of S_{k+1}. Combining the two cases, we have $k+1$ total ways of tiling S_{k+1}, completing the inductive step.

(b) Take two copies of S_{10000} and one copy of S_{2013}. Use them to construct the shape T shown in the diagram at right. (Some squares have been labeled X_i, Y_i, or Z_i, for use in our proof below.)

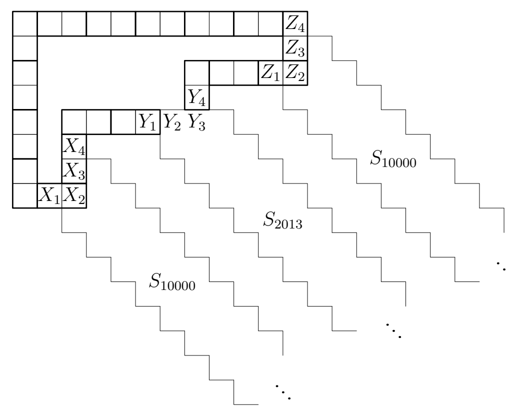


The area of T is

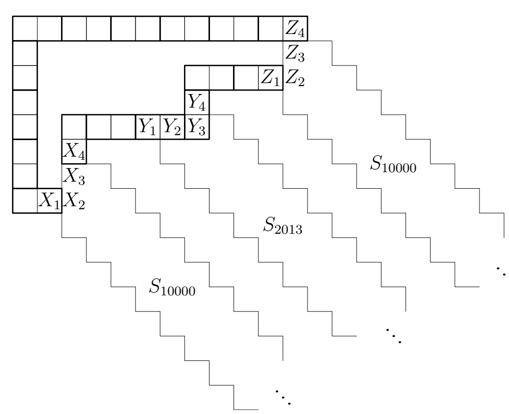
$$30 + 2(2 \cdot 10000) + 2 \cdot 2013 = 44056,$$

which is less than 50000. We now show that T has exactly 100002013 domino-tilings, by showing that all domino-tilings fall into one of the following two cases.

Case 1: The same domino covers X_1 and X_2.
There are now an odd number of remaining squares in the bottom left copy of S_{10000}, so X_3 and X_4 are also covered by the same domino. This forces the dominos as shown to the right. Both copies of S_{10000} have had their two top left squares (X_2, X_3 and Z_2, Z_3) already used, meaning that what remains (for each) is B_{10000} and each has only one way to be domino-tiled. In contrast, the copy of S_{2013} is intact, so it has 2013 ways to be domino-tiled. Thus there are 2013 domino-tilings in this case.



Case 2: Different dominoes cover X_1 and X_2.
Then X_3 and X_4 cannot be covered by the same domino, since this would leave an odd number of squares in S_{10000}. This forces the dominos as shown to the right. Both copies of S_{10000} are intact, meaning they each have 10000 ways of being domino-tiled, and the tilings of each copy of S_{10000} are independent of each other. On the other hand, the copy of S_{2013} has both squares Y_2, Y_3 already used, so what remains is B_{2013}, and this has only one domino-tiling. So there are $10000 \cdot 1 \cdot 10000 = 10000^2$ domino-tilings in this case.



Adding up the two cases, we get $2013 + 10000^2 = 100002013$ total domino-tilings of T, as required.