USA Mathematical Talent Search
Round 3 Solutions
Year 30 — Academic Year 2018–2019
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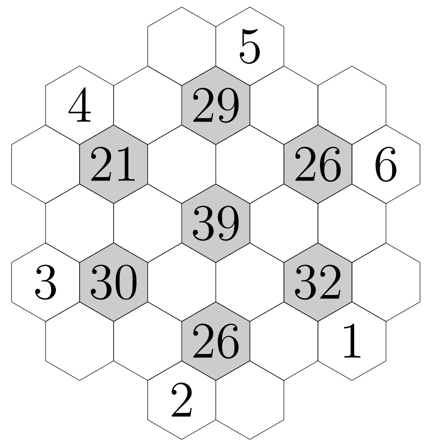
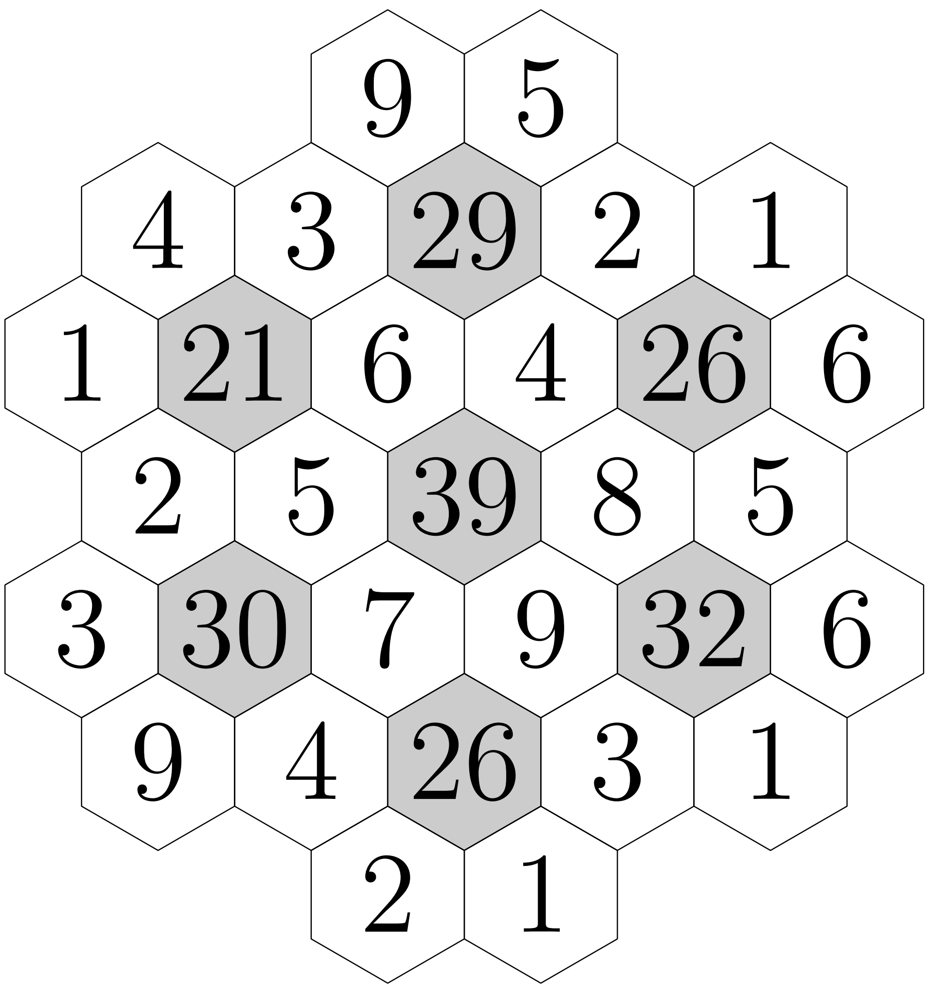
1/3/30 Fill in each white hexagon with a positive digit from 1 to 9. Some digits have been given to you. Each of the seven gray hexagons touches six hexagons; these six hexagons must contain six distinct digits, and the sum of these six digits must equal the number inside the gray hexagon.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



Solution



2/3/30. Lizzie writes a list of fractions as follows. First, she writes $\frac{1}{1}$, the only fraction whose numerator and denominator add to 2. Then she writes the two fractions whose numerator and denominator add to 3, in increasing order of denominator. Then she writes the three fractions whose numerator and denominator sum to 4 in increasing order of denominator. She continues in this way until she has written all the fractions whose numerator and denominator sum to at most 1000. So Lizzie's list looks like:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \dots, \frac{1}{999}.$$

Let p_k be the product of the first k fractions in Lizzie's list. Find, with proof, the value of $p_1 + p_2 + \cdots + p_{499500}$.

Solution

To get started we will list out a few terms p_k and see what we notice. We have

k	p_k
1	1
2	2
3	1
4	3
5	3
6	1
7	4
8	6
9	4
10	1

These numbers are all rows of Pascal's triangle. Notice that 1, 2, 1 is the second row of Pascal's triangle. Then, 1, 3, 3, 1 is the third row and 1, 4, 6, 4, 1 is the fourth row. (There is some overlap in the use of 1s, so we'll have to be careful about that later!) Let's see if we can prove that we will continue to get rows of Pascal's triangle.

If we look at all the fractions with numerator and denominator that sum to $m + 1$, they are:

$$\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}, \dots, \frac{2}{m-1}, \frac{1}{m}.$$

First, note the the product of all of these numbers is 1. This means that $p_t = 1$ for every triangular number t. For example,

$$p_1 = \frac{1}{1} = 1$$

$$p_3 = p_1 \cdot \frac{2}{1} \cdot \frac{1}{2} = 1$$

$$p_6 = p_3 \cdot \frac{3}{1} \cdot \frac{2}{2} \cdot \frac{1}{3} = 1$$

So, let t be the $(m-1)$st triangular number. Then $p_t = 1$, and using the list of fractions above, we have

$$\begin{aligned}
p_{t+1} &= p_t \cdot \frac{m}{1} = m = \binom{m}{1} \\
p_{t+2} &= p_{t+1} \cdot \frac{m-1}{2} = \frac{m(m-1)}{1 \cdot 2} = \binom{m}{2} \\
p_{t+3} &= p_{t+2} \cdot \frac{m-2}{3} = \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} = \binom{m}{3} \\
&\vdots \\
p_{t+m} &= \binom{m}{m} = 1
\end{aligned}$$

Taking this all together, since 499500 is the 999th triangle number, we can rewrite our sum as

$$\binom{1}{1} + \left(\binom{2}{1} + \binom{2}{2}\right) + \left(\binom{3}{1} + \binom{3}{2} + \binom{3}{3}\right) + \left(\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}\right) + \\ + \cdots + \left(\binom{999}{1} + \binom{999}{2} + \cdots + \binom{999}{999}\right)$$

Since $\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n - 1$, we can rewrite this sum as

$$(2-1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{999} - 1).$$

Rearranging, we have

$$(2 + 2^2 + \cdots + 2^{999}) - 999.$$

Then $2 + 2^2 + \cdots + 2^{999} = 2^{1000} - 2$, so our sum is $\boxed{2^{1000} - 1001}$.

3/3/30. Cyclic quadrilateral $ABCD$ has $AC \perp BD$, $AB + CD = 12$, and $BC + AD = 13$. Find the greatest possible area for $ABCD$.

Solution

Let a, b, c, and d be the lengths of the sides in order and let x and y denote the lengths of the diagonals. Note that the two side conditions rewrite as $a + c = 12$ and $b + d = 13$. Furthermore, since the diagonals are perpendicular to each other, we have $a^2 + c^2 = b^2 + d^2$. (This can be proven thorough usage of the Pythagorean Theorem.) To invoke the condition that $ABCD$ is cyclic, we note that by Ptolemy's Theorem,

$$ac + bd = xy = 2\left(\frac{1}{2}xy\right) = 2K,$$

where K is the area of $ABCD$. Thus, maximizing K is equivalent to maximizing $ac + bd$ subject to the conditions above.

Observe that squaring both equations gives $a^2 + c^2 + 2ac = 144$ and $b^2 + d^2 + 2bd = 169$. Since $a^2 + c^2 = b^2 + d^2$, we can subtract these two equations to obtain $bd - ac = \frac{25}{2}$. As a result, $bd + ac = \frac{25}{2} + 2ac$, so maximizing $ac + bd$ thus comes down to maximizing ac. To do this, write $(a + c)^2 - 2ac = 144 - 2ac = b^2 + d^2$. Note that ac is maximized when $b^2 + d^2$ is minimized. But by the Cauchy-Schwarz Inequality

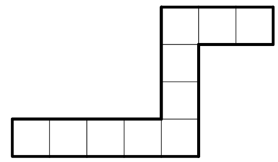
$$2(b^2 + d^2) \geq (b + d)^2 = 169 \quad \Longrightarrow \quad b^2 + d^2 \geq \frac{169}{2}.$$

Therefore $2ac \leq 144 - \frac{169}{2} = \frac{119}{2}$, so

$$bd + ac = \frac{25}{2} + 2ac \leq \frac{25}{2} + \frac{119}{2} = 72$$

and the requested maximum area of $ABCD$ is $\frac{72}{2} = \boxed{36}$. Equality holds when $ABCD$ is an isosceles trapezoid with legs of length 6.5.

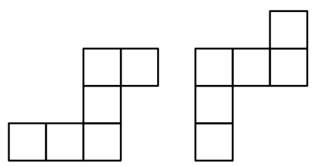
4/3/30. An *eel* is a polyomino formed by a path of unit squares which makes two turns in opposite directions (note that this means the smallest eel has four cells). For example, the polyomino shown at right is an eel. What is the maximum area of a 1000×1000 grid of unit squares that can be covered by eels without overlap?



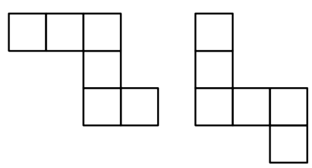
Solution

Suppose that the bottom row of the grid is completely covered by eels and number its squares 1, 2, 3, . . . , 1000 left to right.

There are four possible orientations for an eel. Let type E be the eels that go either right and up, or up and right, as shown below.



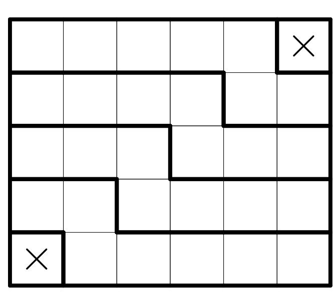
Let type F be the eels such that go down and right, or right and down as shown below.



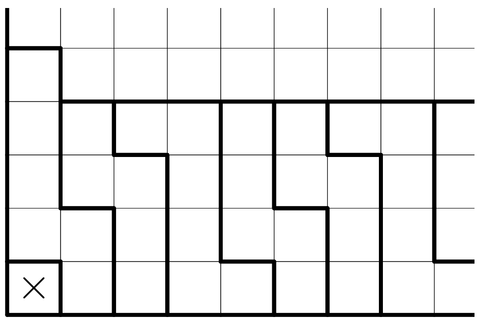
The eel e_1 that covers 1 is of type E. Let e_1 also cover $2, 3, \ldots, a_1$. Then the eel e_2 that covers $a_1 + 1$ is also of type E (because e_1 prevents it from being of type F). Let e_2 also cover $a_1 + 2$, $a_1 + 3$, . . . , a_2, and e_3 be the eel that covers $a_2 + 1$. Then e_3 is also of type E, etc. Eventually, we see that the eel e_k covering 1000 must be of type E, whereas it can only be of type F, a contradiction.

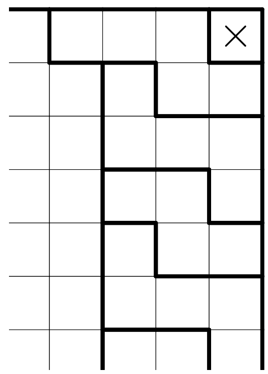
Therefore, the bottom row contains at least one uncovered square. Analogously, so does the top row, the leftmost column and the rightmost column. This amounts to at least two uncovered squares total, and this minimum is attained only if they are at two opposite corners.

How do we construct an example? An initial observation is that an $n \times (n+1)$ rectangle with two opposite corners removed can be covered with eels:

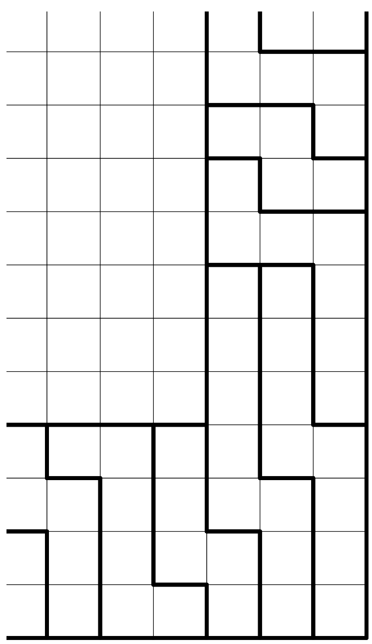


In order to reduce a square to an $n \times (n+1)$ rectangle, we peel off one horizontal strip of width four and one vertical strip of width three:





The two strips join in the bottom right like this.



After removing these two strips from the 1000×1000 square, we're left with a 996×997 rectangle with the opposite corners missing, which we can cover with eels as shown above.

So, the maximum area of a 1000×1000 grid that can be covered by eels without overlap is $1000 \cdot 1000 - 2 = \boxed{999,998}$ square units.

(Problem and solution by Nikolai Beluhov)

5/3/30. The sequence $\{a_n\}$ is defined by $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$,

$$a_n = a_{n-1}^2 + (a_0 a_1 \cdots a_{n-2})^2 .$$

Let k be a positive integer, and let p be a prime factor of a_k. Show that $p > 4(k-1)$.

Solution

Fix a prime p. Let $x_0 = 1$ and $x_i = \frac{a_i}{a_0 a_1 \cdots a_{i-1}} \pmod{p}$, where division is treated as multiplying by a modular inverse. Then for $i \geq 0$,

$$x_{i+1} \equiv x_i + \frac{1}{x_i} \pmod{p},$$

and the sequence terminates as soon as we hit an element that is zero. Moreover, if $x_k \equiv 0 \pmod{p}$, then we know that a_k is divisible by p. Our goal is to show that if $x_j \equiv 0 \pmod{p}$, then j is relatively small compared to p.

Partition the nonzero residues mod p into groups of the form $\{a, -a, a^{-1}, -a^{-1}\}$ where a is a residue mod p. If $p \equiv 3 \pmod{4}$, all such groups have 4 elements except $\{1, -1\}$. If $p \equiv 1 \pmod{4}$, all such groups have 4 elements except $\{1, -1\}$ and $\{b, -b\}$ where $b^2 \equiv -1 \pmod{p}$. So the number of classes is $\lfloor p/4 \rfloor + 1$. Let $m = \lfloor p/4 \rfloor + 1$. Suppose none of $x_0, x_1, x_2, \ldots, x_m$ are zero. By the pigeonhole principle, some two are in the same group of residues, say x_i and x_j. Then either $x_{i+k} \equiv x_{j+k} \pmod{p}$ for all k or $x_{i+k} \equiv -x_{j+k} \pmod{p}$ for all k, so no elements of the infinite sequence $x_0, x_1, x_2, \ldots$ are 0. Therefore, if 0 occurs, it comes within m terms of the sequence.

Suppose now that p divides a_k. Then p does not divide any future term of the sequence, so it is the first term divisible by p. This means x_k exists and is 0. So by the previous paragraph, $k \leq m = \lfloor p/4 \rfloor + 1$. For odd p, this is equivalent to $p \geq 4k - 3$, as desired.

For $p = 2$, note that $a_2 = 2^2 + 1 = 5$, and every term after this is constructed by a sum of the form:

$$(\text{odd number})^2 + (\text{even number})^2$$

since a_1 is even. Thus, no term after a_1 can be divisible by 2.