USA Mathematical Talent Search

Round 2 Problems

Year 24 — Academic Year 2012–2013

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by Monday, November 26, 2012, via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on November 26
   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before November 26.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.
7. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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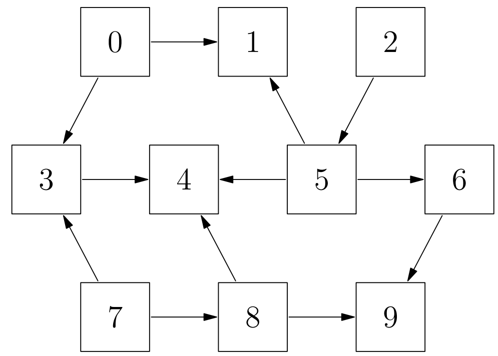
Each problem is worth 5 points.

1/2/24. Fill in each of the ten boxes with a 3-digit number so that the following conditions are satisfied.



1. Every number has three distinct digits that sum to 15. 0 may not be a leading digit. One digit of each number has been given to you.
2. No two numbers in any pair of boxes use the same three digits. For example, it is not allowed for two different boxes to have the numbers 456 and 645.
3. Two boxes joined by an arrow must have two numbers that share an equal hundreds digit, tens digit, or ones digit. Also, the smaller number must point to the larger.

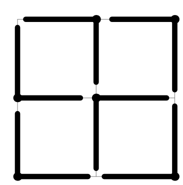
You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/24. Find all triples (a, b, c) of positive integers with $a \leq b \leq c$ such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) = 3.$$

3/2/24. Let $f(x) = x - \frac{1}{x}$, and define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for $n \geq 2$. For each n, there is a minimal degree d_n such that there exist polynomials p and q with $f^n(x) = \frac{p(x)}{q(x)}$ and the degree of q is equal to d_n. Find d_n.

4/2/24. Let n be a positive integer. Consider an $n \times n$ grid of unit squares. How many ways are there to partition the horizontal and vertical unit segments of the grid into $n(n+1)$ pairs so that the following properties are satisfied?



(i) Each pair consists of a horizontal segment and a vertical segment that share a common endpoint, and no segment is in more than one pair.

(ii) No two pairs of the partition contain four segments that all share the same endpoint.

(Pictured above is an example of a valid partition for $n = 2$.)

5/2/24. **(Corrected from an earlier release.)** A unit square $ABCD$ is given in the plane, with O being the intersection of its diagonals. A ray ℓ is drawn from O. Let X be the unique point on ℓ such that $AX + CX = 2$, and let Y be the point on ℓ such that $BY + DY = 2$. Let Z be the midpoint of $\overline{XY}$, with $Z = X$ if X and Y coincide. Find, with proof, the minimum value of the length of OZ. [1]

[1] In the earlier version, the final sentence read "Find, with proof, the minimum value of $|OZ|$," which was inconsistent with how the rest of the problem statement indicated lengths of segments.

Larger diagram for Problem 1/2/24.