USA Mathematical Talent Search

Round 3 Problems

Year 25 — Academic Year 2013–2014

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1 , you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 6, 2014, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 6

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before January 6.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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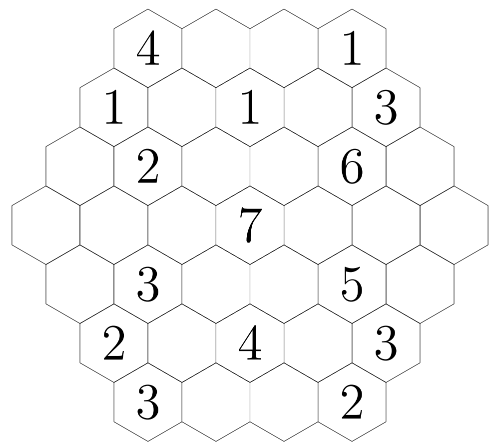
Each problem is worth 5 points.

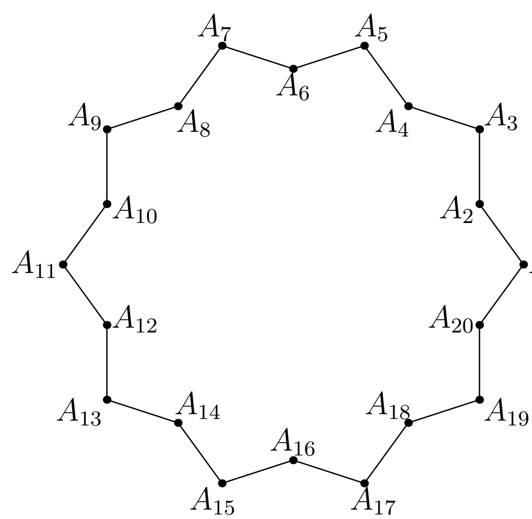
1/3/25. In the grid shown, fill in each empty space with a number, such that after the grid is completely filled in, the number in each space is equal to the smallest positive integer that does not appear in any of the touching spaces. (A pair of spaces is considered to touch if they both share a vertex.)



You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/25. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that $a_k a_{k+2} = a_{k+1} + 1$ for all positive integers k. If a_1 and a_2 are positive integers, find the maximum possible value of a_{2014}.

3/3/25. Let $A_1A_2A_3 \ldots A_{20}$ be a 20-sided polygon P in the plane, where all of the side lengths of P are equal, the interior angle at A_i measures 108 degrees for all odd i, and the interior angle at A_i measures 216 degrees for all even i. Prove that the lines $A_2A_8, A_4A_{10}, A_5A_{13}, A_6A_{16}$, and A_7A_{19} all intersect at the same point.



4/3/25. An infinite sequence $(a_0, a_1, a_2, \ldots)$ of positive integers is called a *ribbon* if the sum of any eight consecutive terms is at most 16; that is, for all $i \geq 0$,

$$a_i + a_{i+1} + \cdots + a_{i+7} \leq 16.$$

A positive integer m is called a *cut size* if every ribbon contains a set of consecutive elements that sum to m; that is, given any ribbon $(a_0, a_1, a_2, \ldots)$, there exist nonnegative integers $k \leq \ell$ such that

$$a_k + a_{k+1} + \cdots + a_\ell = m.$$

Find, with proof, all cut sizes, or prove that none exist.

5/3/25. For any positive integer $b \geq 2$, we write the base-b numbers as follows:

$$(d_k d_{k-1} \ldots d_0)_b = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_1 b^1 + d_0 b^0,$$

where each digit d_i is a member of the set $S = \{0, 1, 2, \ldots, b-1\}$ and either $d_k \neq 0$ or $k = 0$. There is a unique way to write any nonnegative integer in the above form.

If we select the digits from a different set S instead, we may obtain new representations of all positive integers or, in some cases, all integers. For example, if $b = 3$ and the digits are selected from $S = \{-1, 0, 1\}$, we obtain a way to uniquely represent all integers, known as the *balanced ternary* representation. As further examples, the balanced ternary representation of the numbers 5, -3, and 25 are:

$$5 = (1\ {-1}\ {-1})_3, \qquad -3 = (-1\ 0)_3, \qquad 25 = (1\ 0\ {-1}\ 1)_3.$$

However, not all digit sets can represent all integers. If $b = 3$ and $S = \{-2, 0, 2\}$, then no odd number can be represented. Also, if $b = 3$ and $S = \{0, 1, 2\}$ as in the usual base-3 representation, then no negative number can be represented.

Given a set S of four integers, one of which is 0, call S a *4-basis* if every integer n has at least one representation in the form

$$n = (d_k d_{k-1} \ldots d_0)_4 = d_k 4^k + d_{k-1} 4^{k-1} + \cdots + d_1 4^1 + d_0 4^0,$$

where $d_k, d_{k-1}, \ldots, d_0$ are all elements of S and either $d_k \neq 0$ or $k = 0$.

(a) Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a+2\}$ is not a 4-basis.

(b) Show that there are infinitely many integers a such that $\{-1, 0, 1, 4a+2\}$ is a 4-basis.