USA Mathematical Talent Search

Round 1 Problems

Year 30 — Academic Year 2018-2019

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by October 15, 2018, via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on October 15, 2018
   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 15.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.
7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/1/30. Fill in each space of the grid with one of the numbers $1, 2, \ldots, 30$, using each number once. For $1 \leq n \leq 29$, the two spaces containing n and $n+1$ must be in either the same row or the same column. Some numbers have been given to you.

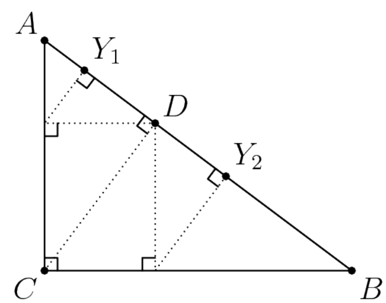
You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

29					
	19			17	
13			21		8
	4		15		24
10				26	

2/1/30. Let $n > 1$ be an integer. There are n orangutoads, conveniently numbered $1, 2, \ldots, n$, each sitting at an integer position on the number line. They take turns moving in the order $1, 2, \ldots, n$, and then going back to 1 to start the process over; they stop if any orangutoad is ever unable to move. To move, an orangutoad chooses another orangutoad who is at least 2 units away from her and hops towards them by a distance of 1 unit. (Multiple orangutoads can be at the same position.) Show that eventually some orangutoad will be unable to move.

3/1/30. Find, with proof, all pairs of positive integers (n, d) with the following property: for every integer S, there exists a unique non-decreasing sequence of n integers $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = S$ and $a_n - a_1 = d$.

4/1/30. Right triangle $\triangle ABC$ has $\angle C = 90°$. A fly is trapped inside $\triangle ABC$. It starts at point D, the foot of the altitude from C to $\overline{AB}$, and then makes a (finite) sequence of moves. In each move, it flies in a direction parallel to either $\overline{AC}$ or $\overline{BC}$; upon reaching a leg of the triangle, it then flies to a point on $\overline{AB}$ in a direction parallel to $\overline{CD}$. For example, on its first move, the fly can move to either of the points Y_1 or Y_2, as shown.



Let P and Q be distinct points on $\overline{AB}$. Show that the fly can reach some point on $\overline{PQ}$.

5/1/30. A positive integer is called *uphill* if the digits in its decimal representation form a non-decreasing sequence from left to right. That is, a number with decimal representation $\overline{a_1 a_2 \ldots a_d}$ is uphill if $a_i \leq a_{i+1}$ for all i. (All single-digit integers are uphill.)

Given a positive integer n, let $f(n)$ be the smallest nonnegative integer m such that $n+m$ is uphill. For example, $f(520) = 35$ and $f(169) = 0$. Find, with proof, the value of

$$f(1) - f(2) + f(3) - f(4) + \cdots + f(10^{2018} - 1).$$

Problems by Kevin Ren, Michael Tang, and USAMTS Staff.

Round 1 Solutions must be submitted by **October 15, 2018**.

Please visit **http://www.usamts.org** for details about solution submission.