USA Mathematical Talent Search

Round 1 Problems

Year 33 — Academic Year 2021-2022

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **October 18, 2021** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on October 18, 2021.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 18, 2021.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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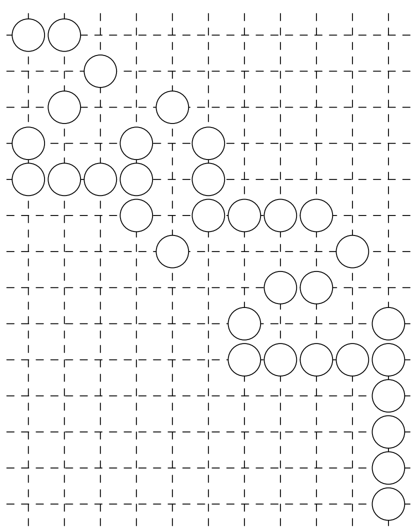
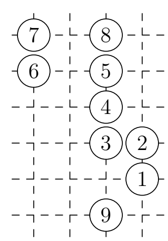
Each problem is worth 5 points.

1/1/33. 33 counters are shown in the left grid below. Choose a counter to start at and remove it from the grid. At each subsequent step, choose a direction (up, down, left, or right), move along the grid line from your current position to the nearest counter in that direction, and remove that counter. You cannot choose a direction that reverses your previous one (e.g., left then right is not allowed). Your goal is to pick up all 33 counters in a single sequence of steps. When you find the right sequence, write the numbers 1 to 33 on the counters so that N is written on the N^{th} counter you removed.

A smaller example of a solved grid is shown to the right below. (Note that the final move from 8 to 9 is possible because counters 3, 4, and 5 have been removed in earlier steps.)

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)





2/1/33. Find, with proof, the minimum positive integer n with the following property: for any coloring of the integers $\{1, 2, \ldots, n\}$ using the colors red and blue (that is, assigning the color "red" or "blue" to each integer in the set), there exist distinct integers a, b, c between 1 and n, inclusive, all of the same color, such that $2a + b = c$.

3/1/33. Let S be a subset of $\{1, 2, \ldots, 500\}$ such that no two distinct elements of S have a product that is a perfect square. Find, with proof, the maximum possible number of elements in S.

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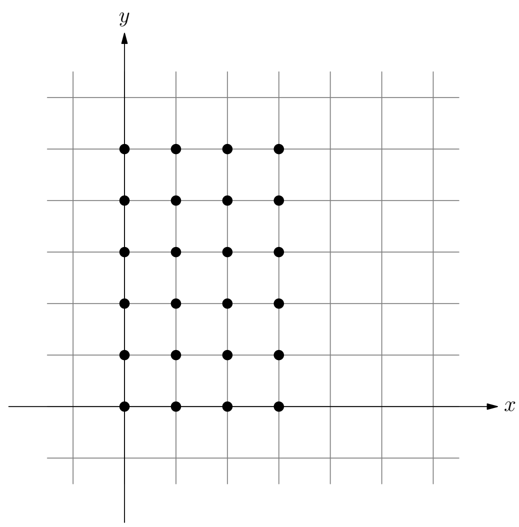
4/1/33. Let m, n, k be positive integers such that $k \le mn$. Let S be the set consisting of the $(m+1)$-by-$(n+1)$ rectangular array of points on the Cartesian plane with coordinates (i, j) where i, j are integers satisfying $0 \le i \le m$ and $0 \le j \le n$. The diagram below shows the example where $m = 3$ and $n = 5$, with the points of S indicated by black dots:



Prove that there exist points A, B, C in S such that the area of $\triangle ABC$ is $\frac{k}{2}$.

5/1/33. Define a sequence of positive rational numbers $x_0, x_1, x_2, x_3, \ldots$ by $x_0 = 2$, $x_1 = 3$, and for all $n \ge 2$,

$$x_n = \frac{x_{n-1}^2 + 5}{x_{n-2}}.$$

(a) Prove that x_n is an integer for all $n \ge 0$.

(b) Prove that if x_n is prime, then either $n = 0$ or $n = 2^k$ for some integer $k \ge 0$.

Problems by Valentin Vornicu, Carl Yerger, and USAMTS Staff.

Round 1 Solutions must be submitted by **October 18, 2021**.

Please visit **http://www.usamts.org** for details about solution submission.