USA Mathematical Talent Search

Round 4 Problems

Year 20 — Academic Year 2008–2009

www.usamts.org

Important! New deadline for web submission: 3 PM Eastern / Noon Pacific

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by March 9, 2009, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on March 9

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before the deadline day.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)

7. Round 4 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/4/20. Consider a sequence $\{a_n\}$ with $a_1 = 2$ and $a_n = \dfrac{a_{n-1}^2}{a_{n-2}}$ for all $n \geq 3$. If we know that a_2 and a_5 are positive integers and $a_5 \leq 2009$, then what are the possible values of a_5?

2/4/20. There are k mathematicians at a conference. For each integer n from 0 to 10, inclusive, there is a group of 5 mathematicians such that exactly n pairs of those 5 mathematicians are friends. Find (with proof) the smallest possible value of k.

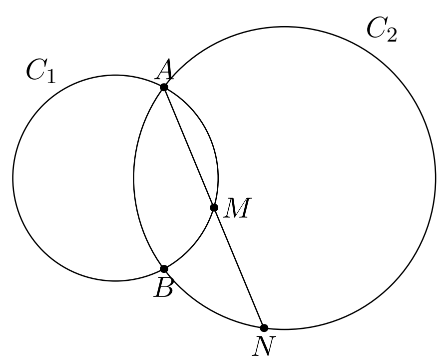
3/4/20. A particle is currently at the point $(0, 3.5)$ on the plane and is moving towards the origin. When the particle hits a lattice point (a point with integer coordinates), it turns with equal probability 45° to the left or to the right from its current course. Find the probability that the particle reaches the x-axis before hitting the line $y = 6$.

4/4/20. Find, with proof, all functions f defined on nonnegative integers taking nonnegative integer values such that

$$f(f(m) + f(n)) = m + n$$

for all nonnegative integers m, n.

5/4/20. A circle C_1 with radius 17 intersects a circle C_2 with radius 25 at points A and B. The distance between the centers of the circles is 28. Let N be a point on circle C_2 such that the midpoint M of chord AN lies on circle C_1. Find the length of AN.



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Mail deadline: Postmarked on or before due date
Please visit **http://www.usamts.org** for details about solution submission.