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Solutions to Problem 5/4/17
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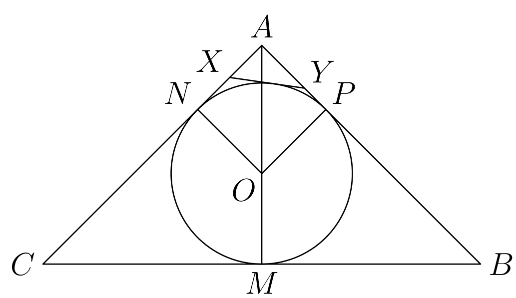
5/4/17. Sphere $\mathcal{S}$ is inscribed in cone $\mathcal{C}$. The height of $\mathcal{C}$ equals its radius, and both equal $12 + 12\sqrt{2}$. Let the vertex of the cone be A and the center of the sphere be B. Plane $\mathcal{P}$ is tangent to $\mathcal{S}$ and intersects segment $\overline{AB}$. X is the point on the intersection of $\mathcal{P}$ and $\mathcal{C}$ closest to A. Given that $AX = 6$, find the area of the region of $\mathcal{P}$ enclosed by the intersection of $\mathcal{C}$ and $\mathcal{P}$.

Credit This problem was proposed by Richard Rusczyk.

Comments As with most problems in three-dimensional geometry, a solution can be found by considering relevant cross-sections of the figure. Many students made incorrect assumptions about the figure, such as assuming that the altitude of the cone from A intersects the ellipse at its center. An accurately drawn figure helps prevent such errors. *Solutions edited by Naoki Sato.*

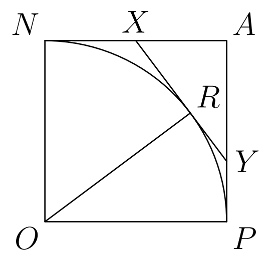
Solution 1 by: Eric Chang (11/CA)



Let Y be the furthest point on the intersection of $\mathcal{P}$ and $\mathcal{S}$ from A. Let O be the center of sphere $\mathcal{S}$. Let BC be a diameter of the base of cone $\mathcal{C}$. Let N and P be the points where the sphere $\mathcal{S}$ is tangent to AB and AC, respectively. Finally, let M be the center of the base of cone $\mathcal{C}$.

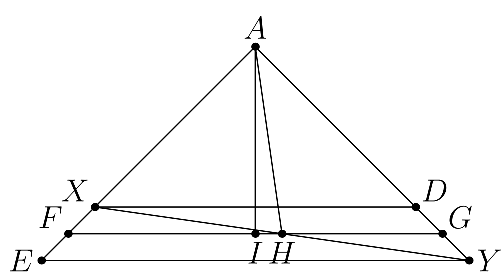
First of all, we see that the intersection of plane $\mathcal{P}$ and cone $\mathcal{C}$ will be an ellipse, which we call $\mathcal{E}$. If we take a cross section of the cone and the sphere along plane ABC, we get the above picture. Since the area of an ellipse is πab, with a and b as the semi-major and semi-minor axis, respectively, we can solve for the area if we can find the length of the major and minor axis. It is obvious that XY is the major axis of ellipse $\mathcal{E}$ since it is the longest line in the ellipse. Also, by symmetry M is the midpoint of side BC, and A, O amd M are collinear.

Because $\angle OPA$, $\angle ONA$, and $\angle NAP$ are all right angles, $\angle NOP$ must be a right angle also. Since all segments tangent to a circle from the same point have the same length, $AN = AP$ and we see that $ANOP$ is a square. Let r be the radius of $\mathcal{S}$, then we see from the picture $AM = r\sqrt{2} + r = 12 + 12\sqrt{2}$, since it is the height of the cone. Solving we get $r = 12$. Now we will consider square $ONAP$, reproduced below:

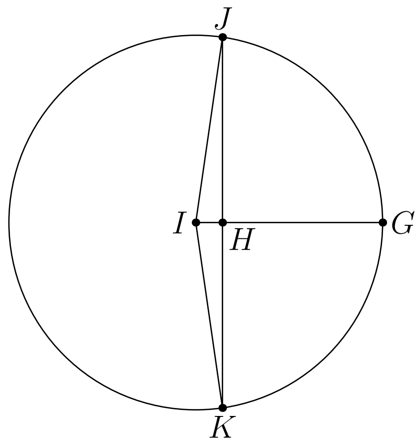


Let R be the point of tangency of XY to the circle centered at O. Since all sides of a square are congruent, they are all equal to 12, therefore, $NX = 12 - 6 = 6$. Since all segments tangent to the circle from the same point are congruent, $XR = 6$ and $RY = PY$. Labeling $AY = x$, we see that $PY = 12 - x$ and $XY = 18 - x$. Also, since AXY is a right triangle, $XY^2 = x^2 + 6^2$. Equating the two expressions for XY and solving for x, we get $x = 8$, and therefore, $AY = 8$ and $XY = 10$.

We will construct the diagram below in the following paragraph. Draw a line through X parallel to BC, and call D its point of intersection with AB, and then draw a line through Y parallel to BC also, and the point E will be its intersection with AC. Now, since these lines are parallel to the base, by similar triangles we have $AX = AD = 6$ and $AE = AY = 8$, which implies $XE = DY = 2$. If we draw line FG parallel to XD and EY and go through the midpoint of XE, it will also go through the midpoint of XY and DY because parallel lines cut all transversals in the same ratio. Call H its intersection with XY, the the minor axis will pass through this point perpendicular to XY.



Now draw the segment AI, where I is the midpoint of FG, then by a property of isosceles triangles, AI is perpendicular to FG. As a result, we can find HI by using the Pythagorean theorem on triangle AHI. Since XAY is a right triangle, by a well-known theorem, the segment AH will be congruent to XH and HY, therefore $AH = 5$. Also, using the 45-45-90 triangle AFI, we find that $AF = AX + XF = 6 + 1 = 7$. Therefore, $AI = \frac{7\sqrt{2}}{2}$ and $HI^2 = 5^2 - \left(\frac{7\sqrt{2}}{2}\right)^2 = \frac{1}{2}$, so $HI = \frac{\sqrt{2}}{2}$.



Now we take a top view of the cross section of the cone at the circle centered at I. Let JK be the minor axis. We can see that the radius of the circle is $\frac{7\sqrt{2}}{2}$ from previous calculations, and since HI is perependicular to JK, which is JH, using the Pythagorean theorem again:

$$JH^2 = \left(\frac{7\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 24,$$

so $JH = \sqrt{24} = 2\sqrt{6}$. Now we can plug in $2\sqrt{6}$ and $10/2 = 5$ into out formula πab. As a result, the area of ellipse $\mathcal{E}$ is $10\pi\sqrt{6}$.

Note: There is a beautiful solution using Dandelin spheres. Not only does this approach solve the problem in a nice, synthetic way, it also explains why the cross-section is an ellipse. See

`http://www.artofproblemsolving.com/Community/AoPS_Y_MJ_Transcripts.php?mj_id=128`

for a transcript of this solution.